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Iterations of Transformations on the Unit Interval: Approach to a Periodic Attractor

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We consider a transformation of the unit interval which exhibits a stable periodic attractor and whose topological entropy is positive. We show that the dynamics leading to this attractor has a statistical character.

KEY WORDS: Iteration of transformations; attractors; periodic orbits; Cantor sets; invariant measure; topological entropy.

1. INTRODUCTION

We consider noninvertible transformations of the unit interval into itself

$$T(x, \tilde{\lambda}): \quad x' = f(x, \tilde{\lambda}) \tag{1}$$

where $\tilde{\lambda}$ is a set of parameters. A well-known example of such transformations has been studied by Metropolis *et al.*[1] (MSS), in which $\tilde{\lambda}$ reduces to a single parameter λ and $f(x, \tilde{\lambda}) \equiv \lambda f(x)$, $f(x)$ having only one maximum inside the interval, and obeying a set of fairly general constraints (see Ref. 1). The semigroup of transformations $T^n(x, \tilde{\lambda})$, where

$$\underbrace{T^n = T \circ T \circ \cdots \circ T}_{n \text{ times}}$$

is a dynamical system which associates to any point x of $[0, 1]$ an "orbit," i.e., the set $\{x, T(x),..., T^n(x)\}$. An orbit is said to be periodic of period k if it contains k points, and if $T^k(x) = x$ for an orbit point x. This orbit is said to be "stable" or to be an "attractor" if the transformation $T^*(x, \tilde{\lambda})$ is contracting in a neighborhood of any orbit point.

The interest for the physicist of transformation (1) is that it may generate "complicated dynamics," in the words of May,[2] even if $f(x)$ is quite simple

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(for instance, quadratic). According to the values of the parameters in $\{\tilde{\lambda}\}$ which control the bifurcations of the dynamical system, the orbit of a point, after a large number of iterations, may either converge toward a periodic attractor or wander erratically in a subset of [0, 1]. In the latter case, if there exists an invariant, nonatomic measure whose support has positive Lebesgue measure, this support may be called an "aperiodic attractor." It is important to remark that the character partly deterministic, partly stochastic of the dynamics seems to be the rule in the systems encountered in physics (or in biology): let us mention Hamiltonian nonintegrable systems and dissipative systems such as the Lorenz[5] and Curry[6] models of convection and Spiegel oscillators.[7]

Let us now recall some typical properties of the quadratic transformation $T \equiv \lambda x(1-x)$ (these properties are shared by a subclass of the MSS transformations under additional conditions which need not be specified here). There are two domains of λ values where the dynamics is essentially different. In the first one ($[\lambda_0, \lambda_c[$), there exists a finite number of periodic orbits and, among them, a unique periodic attractor with period $k = 2^N$, N going to infinity when $\lambda \to \lambda_c$ (see for instance, Ref. 2–4). In this domain the dynamics may be considered as "simple." In the second domain ($]\lambda_c, \lambda_{max}]$) there is an infinite set of periodic orbits and the topological entropy h_T is positive. The positiveness of h_T may be looked at as a characteristic feature (a definition?) of "complicated dynamics," and the $]\lambda_c, \lambda_{max}]$ domain is frequently called "chaotic." Actually, an aperiodic attractor is to be found for an infinite set of λ values. Unfortunately, these attractors are not structurally stable (they are destroyed by an arbitrary small variation of λ). Moreover, there are infinitely many values of λ (probably forming a dense set in $[\lambda_0, \lambda_{max}]$) where there exists a periodic and structurally stable attractor. For these λ values, the asymptotic dynamics leads the orbit toward a limit cycle (an obviously "simple" situation), provided the Lebesgue measure of the basin of the periodic attractor is unity.

For simplicity, we shall consider transformations $T(x, \tilde{\lambda})$ for which there exists a unique periodic attractor. This is ensured by assuming the negativity of its Schwarzian derivative with respect to x (see Ref. 12). In order to rule out the possibility of complicated asymptotic dynamics in the range of $\tilde{\lambda}$ values where the transformation exhibits a periodic attractor, we must first answer the question: can we find nontrivial transformations such that the Lebesgue measure of the periodic attractor's basin is unity? It is a rather difficult problem to find out the least stringent conditions which ensure that $T(x, \tilde{\lambda})$ has this property. In connection with this problem, we mention the Lasota theorem,[10] which may be stated as: if T is a continuous mapping of the unit interval into itself which has a periodic orbit of period 3, then T admits an invariant, nonatomic measure. However, the support of this measure may have zero Lebesgue measure. It is not our purpose in this paper to determine the above-mentioned conditions. We only need that they are obeyed by the two transformations which we explicitly consider here, namely the quadratic and the trapezoidal transformations. In the first case, Smale[11] showed that the limit set of this system consists of the periodic orbit and of a Cantor set with zero Lebesgue measure. In the second case, we show that the Lebesgue measure of the periodic attractor's basin is unity. We also show, through a straightforward argument in the Appendix, the existence of an open class of transformations exhibiting this property.

Now our problem is the following. Considering a transformation $T(x, \tilde{\lambda})$ of the above type, i.e., whose the unique attracting object is a periodic orbit but whose topological entropy is positive, in what sense can we consider that the dynamics has a stochastic character? Clearly, the complexity of the dynamics can only manifest itself in the approach of the attractor (transitory regime). The purpose of this paper is to give some insight into that approach, and to show that it is indeed of stochastic character. More precisely, we show that, assuming initial points x randomly chosen in $[0, 1]$, the dynamics generated by iterating $T(x, \tilde{\lambda})$ is very similar to a random walk in $[0, 1]$ in the presence of an absorbing interval. Moreover, the detailed analysis of the random approach of the attractor manifests the Cantor structure of the set of nonwandering points of the transformation.

2. SOME NOTATIONS AND DEFINITIONS

2.1. Inverses of Points and Intervals

The graph of a typical transformation $T(x, \tilde{\lambda})$ is shown in Fig. 1. A point of $[0, 1]$ has preimages or "inverses" if it lies on the left of image a of the maximum (see the geometrical construction of the inverses of a point x in Fig. 1). Defining $T^{-k}(x) = \{y | T^k(y) = x\}$, we shall say that $y \in T^{-k}(x)$ is an inverse of x "of order k." Considering an interval $[\xi]$, we define in the same way $T^{-n}([\xi]) = \{x | T^n(x) \in [\xi]\}$. We observe that

$$T^{-n}([\xi]) = \bigcup_{i=1}^{m} I_{n,i}([\xi])$$

with $m \leqslant 2^n$, and where $I_{n,i}$ are intervals generated by successive steps according to

$$I_{p+1,\{j\}} = T^{-1}(I_{p,k})$$

$I_{p+1,\{j\}}$ being a set of 0, 1, or 2 intervals whose images by T are $I_{p,k}$. It is easily seen by induction that the $I_{n,i}$ are disjoint intervals: they will be called "inverses of $[\xi]$ of order n."

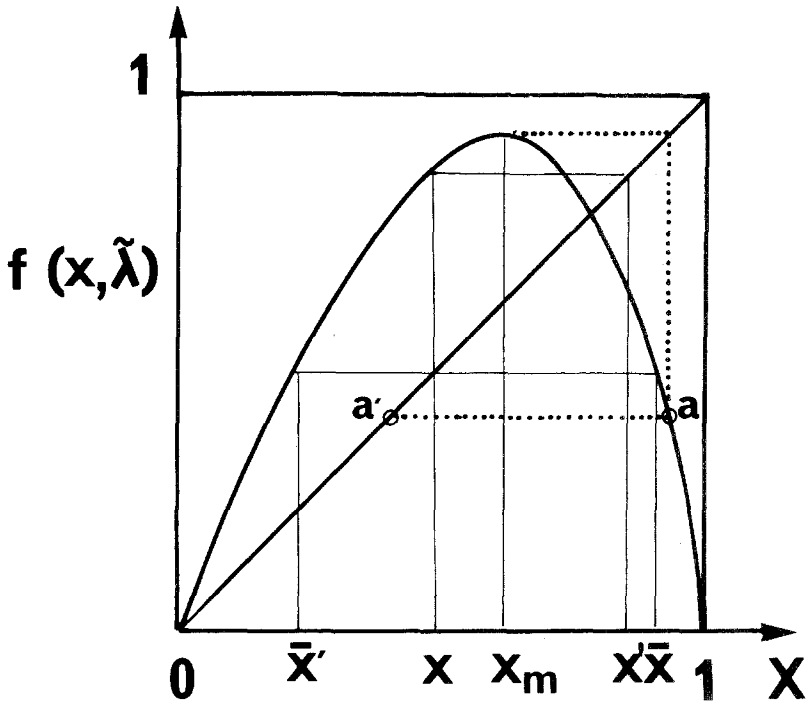

Fig. 1. Graph of a typical $T(x, \tilde{\lambda})$ transformation.

2.2. Invariant Intervals. "Trapping Interval"

Obviously [0, 1] is invariant under a transformation $T(x, \tilde{\lambda})$ whose unique maximum with respect to x is smaller than one. Let x_m be the point where T is maximum and a and a' be respectively $T(x_m)$ and $T^2(x_m)$. A smaller invariant interval is $[\Delta] = [a, a']$, and the orbit of any point of the complement of $[\Delta]$ reaches $[\Delta]$ after a finite number of iterations (except for points 0 and 1). Δ will be the length of $[\Delta]$, a notation which will be used throughout this paper for any interval.

Let us now consider a periodic attractor of period k generated by a "saddle node bifurcation." Varying the relevant parameters in $\tilde{\lambda}$, $f^k(x, \tilde{\lambda})$ comes into contact with the diagonal for some definite value $\tilde{\lambda}_0$, then crosses the diagonal, creating k pairs of fixed points alternately stable and unstable. This type of bifurcations is pictured in Fig. 2. Considering a particular pair of fixed points (p, q) together with point (q') which is a nearest inverse of q on the left side of p, we see that $[\delta] = [q, q']$ is an invariant interval for $T^k(x, \tilde{\lambda})$ analogous to [0, 1] for $T(x, \tilde{\lambda})$. Clearly $[q, q']$ is a connected component of the basin of attraction of p. Any point $x \in]q, q']$ approaches p monotonically after a finite number of iterations (in the sense that the distance between p and

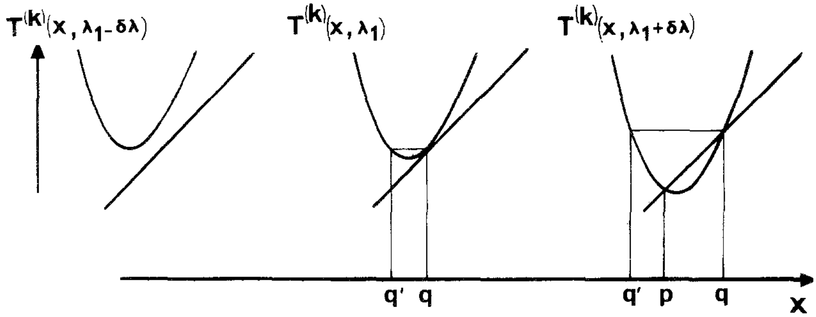

Fig. 2. A saddle-node bifurcation around $\lambda = \tilde{\lambda}_1$.

the successive iterates of x by T^k decreases monotonically). Each point of the attractor is surrounded by such an invariant interval, and the k intervals are trivially obtained from one of them by applying the T transformation k times. When x_m belongs to one of these intervals (which is the case when the Schwarzian derivative of T is negative, and for the trapezoidal transformation), we shall call this interval the "trapping interval" associated with the periodic attractor.

It is natural to distinguish two regimes in the dynamics: in the first one the orbit is still outside $[\delta]$ and may wander erratically in $[\Delta] - [\delta]$ ("transitory stochastic regime"); in the second one ($x \in [\delta]$) the orbit is driven steadily toward the fixed points of $T^{(k)}$ ("deterministic regime"). It is interesting to remark that, as $\tilde{\lambda}$ crosses a bifurcation value, the new-born periodic orbit already appears with a finite trapping interval (see Fig. 2).

Let us now suppose that, for some given $\tilde{\lambda}$, the transformation has a unique periodic attractor, in addition to a finite set of (unstable) fixed and periodic points. In such a situation (encountered for $\lambda < \lambda_c$) where the topological entropy h_T is null, the orbit of almost any initial point will reach the trapping region after a finite number of iterations. In this case the preasymptotic dynamics cannot really be considered as stochastic, and we shall therefore disregard the systems where h_T is not positive. In the following we shall support this point of view by displaying the Cantor-set-like structure of statistical behavior in the preasymptotic regime for $\lambda > \lambda_c$.

3. APPROACH OF A PERIODIC ATTRACTOR

3.1. The Case of the Trapezoidal Application

3.1.1. Preliminaries. This transformation is defined as

$$T_z(x, \tilde{\lambda}): \quad x' = f(x, \tilde{\lambda})$$

when $\tilde{\lambda}$ is the set of three parameters λ, λ', b and

$$f(x,\tilde{\lambda}) = \begin{cases} \lambda' x, & 0 < x < b/\lambda' \\ b, & b/\lambda' < x < 1 - b/\lambda \\ \lambda(1-x), & x > 1 - b/\lambda \end{cases}$$

We first verify that there is no more than one periodic attractor for a given set $\tilde{\lambda}$. Indeed, one of the cycle points, say p, must lie inside the plateau $[\bar{\omega}] = [h', h]$ where the application is contracting. Then, p is the kth image of the plateau (in the case of period k cycle), and it is obviously unique. The same argument shows that $[\bar{\omega}] \subset [\delta]$: If λ and λ' are both larger than unity, the transformation is dilating outside $[\delta]$.

We shall consider first the case $k = 3$, the orbit points being $\{p, a, a'\}$ shown in Fig. 3. The invariant and trapping intervals are respectively $[\Delta] = [a', a]$ and $[\delta] = [q', q]$.

We now show that the basin of the period-3 attractor has Lebesgue measure equal to unity. We first remark that the orbit of almost any point in $[0, 1]/[\Delta]$ reaches $[\Delta]$, and since $[\Delta]$ is invariant by T, we have to show that the measure of the union of the inverses of $[\delta]$ lying in $[\Delta]$ is Δ. As we shall see,

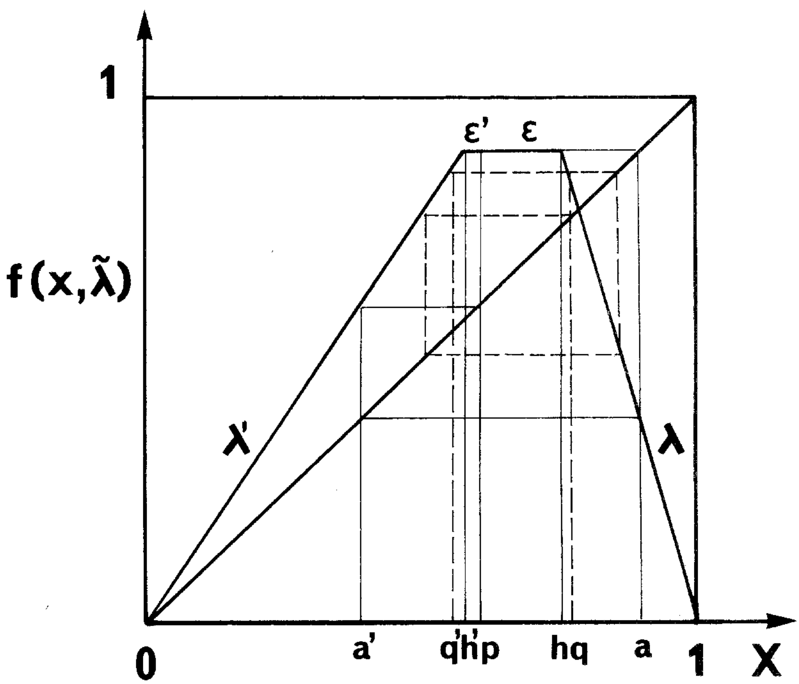

Fig. 3. The trapezoidal transformation $T_z(x, \tilde{\lambda})$.

this interpretation of Δ will be very useful for analyzing the stochastic dynamics in the "transitory regime." However, a difficulty is that some inverses of $[\delta]$ are overlapping (an inverse of $[\delta]$ is to be understood as one of the $I_{n,i}([\delta])$ defined in the previous section).

It proves to be more convenient to consider the inverses of the plateau $[\bar{\omega}]$, which are disjoint, as we shall see. It is not hard to see that the sum of the lengths of $[\bar{\omega}]$ inverses is also Δ. Indeed, there is a particular set of $T^{-(3n)}$ inverses which covers exactly $[\delta]/[\bar{\omega}]$: this is shown in Fig. 4.

Another difficulty is the following: any point in $[\Delta]$ has two inverses, except $\{a\}$, whose inverse is interval $[\bar{\omega}]$. We shall bypass this difficulty by considering, instead of $[\bar{\omega}]$, the union of open intervals $[\tilde{\bar{\omega}}] =]a', p[\cup]p, a[$. Then, given any $x \in [\tilde{\bar{\omega}}]$, the set of its nth inverses $\{T^{-n}(x)\}$ contains a finite number of points, $\forall n$. We shall now prove the following result.

Lemma. The inverses of $[\tilde{\bar{\omega}}]$ are not overlapping.

Indeed, let α and β be two inverses of $[\tilde{\bar{\omega}}]$, respectively of order p and k. If $p = k$, the property is obvious by geometrical construction. If $p \neq k$ $(p > k)$, let $x \in \alpha \cap \beta$. We have $T_z^{(k+1)}(x) = a$. Then, x is an inverse of a, and therefore does not belong to an inverse of $[\tilde{\bar{\omega}}]$.

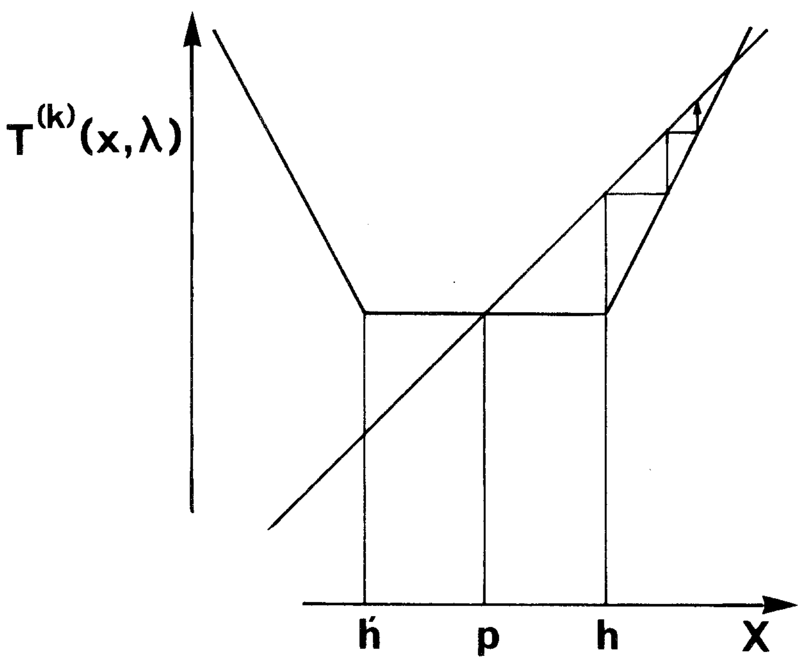

Fig. 4. Successive preimages of the plateau $[h', h]$ covering the trapping interval.

Let us put $F = \text{meas}\{\text{union of all inverses of } [\tilde{\tilde{\omega}}] \text{ lying in } [\Delta]\}$.

Using the above lemma, F may be replaced by the sum of the lengths of $[\tilde{\tilde{\omega}}]$ inverses. We shall give an analytic expression for F (which will be useful in the statistical analysis) and verify that $F = \Delta$. The orbit points $\{p, a, a'\}$ generate a partition of $[\Delta] - \{p, a, a'\}$. Let $[1] =]p, a]$ and $[2] =]a', p[$ be the elements of this partition. A point x in $[\Delta]$ has one or two inverses in $[\Delta]$ according to whether x belongs to $[1]$ or to $[2]$. Now, given any subinterval $[\eta]$ in $[\tilde{\tilde{\omega}}]$, let δ_n be the sum of the measures of all inverses of order $(\cup_i I_{n,i}([\eta]))$ which are found in $[1]$, and Σ_n the corresponding sum for inverses belonging to $[2]$. We have

$$\delta_{n+1} = (\delta_n + \Sigma_n)/\lambda, \qquad \Sigma_{n+1} = \delta_n/\lambda'$$

therefore, the vector X_n with components (δ_n, Σ_n) obeys the recurrence law

$$X_{n+1} = (A/\lambda)X_n$$

where

$$A = \begin{pmatrix} 1 & 1 \\ \mu & 0 \end{pmatrix}, \qquad \mu = \frac{\lambda}{\lambda'}$$

from which $X_n = (A/\lambda)^n X_0$, where X_0 is the vector whose components are the lengths of $([\eta] \cap [1])$ and $([\eta] \cap [2])$. Let us remark that the matrix $\tilde{A} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, obtained by setting $\mu = 1$ in A, is nothing but the "transition matrix" used in symbolic dynamics (cf. for example Ref. 12). $\tilde{A}$ permits one to calculate the number of inverses of any point in $[\Delta]$.

The partition $\{[1], [2]\}$ of $[\Delta]$ is a generating partition for evaluating the topological entropy h_T, and λ_0, the largest value of A, is equal to e^{h_T} [in this case of a period-3 cycle, $\lambda_0 = (1 + \sqrt{5})/2$]. We can now evaluate F

$$F = \left(\mathfrak{U}, \sum_{k=0}^{\infty} \frac{A^n}{\lambda^n} X_0\right)$$

where $X_0 = (\epsilon, \epsilon')$ and $\mathfrak{U} = (1, 1)$.

Now $(\mathfrak{I} - A/\lambda)$ is an invertible matrix except if $\det(I - A/\lambda) = 0$, which happen if $\lambda\lambda' - \lambda' - 1 = 0$, that is, if the trapezium reduces to a triangle ($\epsilon = \epsilon' = 0$). Disregarding this case, we have

$$F = [\mathfrak{U}, (\mathfrak{I} - A/\lambda)^{-1} X_0]$$

which gives

$$F = \frac{\lambda}{\lambda\lambda' - \lambda' - 1} [\epsilon(\lambda' + 1) + \epsilon'\lambda']$$

The length Δ of the invariant interval is

$$\Delta = \lambda'(p - \epsilon') - \lambda[1 - \lambda'(p - \epsilon')]$$

where p, ϵ, ϵ' are known functions of λ, λ', b. There exist two relations connecting λ, λ', p, ϵ, ϵ'. The first one is geometrical: $\lambda'(p - \epsilon') = \lambda[1 - (p + \epsilon)]$, while the second one says that $\{p\}$ is a fixed point of $T_z^{(3)}$: $\lambda\lambda'[1 - \lambda'(p - \epsilon')] = p$. Eliminating ϵ and ϵ' between these two relations, we obtain

$$F = \Delta = 1 - \frac{p(\lambda + 1)}{\lambda\lambda'} = 1 + \left(\lambda - \frac{1}{\lambda}\right)b$$

as expected. In the case of a stable orbit with period $k > 3$, the ingredients of the calculation would be the same. The partition of $[\Delta]$ would contain $k - 1$ elements, X_0 having nonzero components (ϵ and ϵ') on the only two elements adjacent to p [needless to say, inverting the $(k - 1) \times (k - 1)$ matrix $(\mathfrak{I} - A/k)$ is tractable only for moderate values].

3.1.2. Approach of the Periodic Attractor. The aim of this paper is to characterize the random approach of the trapping interval by averaging over a statistical ensemble of initial points. This will be done by considering the set of trapping interval inverses. To be specific, let us consider first the case of the trapezoid application. Let T_t be the triangular transformation with the same slopes λ, λ' as those of the previous trapezium. The orbits generated by iterations of T_t and T_z are the same as long as they do not enter $[\omega]$. When $T_z \to T_t(\epsilon + \epsilon' \to 0)$ in such a way that a k periodic attractor still exists, the T_z dynamics is the same as the T_t dynamics for very long times. At the limit $\epsilon + \epsilon' = 0$ we recall that dynamical system T_t possesses an invariant measure, which is constant in the $k - 1$ intervals delimited by the points of the k orbit. For finite $\epsilon + \epsilon'$ (or finite $[\delta]$) the average lifetime $\langle n \rangle$ of "chaotic dynamics" will be defined as the average number of iterations after which the orbit of an initial point, chosen at random in $[\Delta]$ with uniform probability, reaches $[\delta]$. In the same way, we define the rth moment $\langle n^r \rangle$, and more generally the probability law $P(n)$ of random variable n. The $P(n)$ will be determined by the measures of the inverses of $[\delta]$, and we can find an analytical expression for it in the case of the T_z transformation (and for other suitable polygonal transformations). For the reasons given in the previous section, we prefer to consider the inverses of $[\tilde{\omega}]$. There results a slight change in the above definition of $P(n)$, which is relatively unimportant, because the approach of the stable fixed point in $[\delta]$ by $T_z^{(k)}$ is monotonic. Therefore, the orbits of most of the points in $([\delta] - [\tilde{\omega}])$ reach $[\tilde{\omega}]$ after few iterations of $T_z^{(k)}$.

As was said above, the orbit points of a k periodic attractor delimit a natural partition of $[\Delta]$ in $k - 1$ intervals, and a $(k - 1) \times (k - 1)$ matrix A associated with this partition. The expectation value of random variable n^r

will clearly be expressed in terms of A as

$$\langle n^r \rangle = \frac{1}{\Delta}\left\{\mathfrak{I}, \left(\sum_{p=0}^{\infty} n^r \frac{A^n}{\lambda^n}\right) X_0\right\} \tag{2}$$

with the same notation as in the preceding section; this can be generalized to the general case of a k periodic orbit.

Putting $\lambda = e^{-z}$, we can write

$$\langle n^r \rangle = \frac{1}{\Delta}\left[\mathfrak{I}, \left(\sum n^r e^{nz} A^n\right) X_0\right]$$

$$= \frac{1}{\Delta}\frac{\partial^r}{\partial z^r}[\mathfrak{I}, (\mathfrak{I} - e^z A)^{-1} X_0] \tag{3}$$

$$\langle n^r \rangle = \frac{1}{G}\frac{\partial r}{\partial z^r} G(z, \epsilon, \epsilon')$$

where $G(z, \epsilon, \epsilon') = U(e^{-z}, \epsilon, \epsilon') = \Delta(e^{-z}, \epsilon, \epsilon')$ according to the result of the previous section.

A consequence of Eq. (3) is that $\Delta(e^{-z}, \epsilon, \epsilon')$ plays the role of the generating function of probability law $P(n)$. Let us indeed evaluate the expectation value of e^{-sn}, where s is an arbitrary complex number, and looking at n as a positive real number. We have

$$\langle e^{-sn} \rangle G(z) = G(z) \int_0^\infty e^{-sn} P(n)\, dn = G(z)\left[1 + \sum_{r=1}^{\infty} \frac{(-s)^r}{r!} \langle n^r \rangle\right] = G(z - s)$$

with the help of Eq. (3).

Therefore, $G(z - s)$ appears as the Laplace transform of $G(z)P(n)$, or

$$P(n) = \frac{1}{G(z)} \int_C e^{sn} G(z - s)\, ds \tag{4}$$

where (C) is the Bromwhich contour of the inverse Laplace transform in the complex s plane. Equation (4) can also be written as

$$P(n) = \frac{1}{\Delta(\lambda, \epsilon, \epsilon')} \int_C e^{sn} \Delta(\lambda e^s, \epsilon, \epsilon')\, ds$$

Let us consider, for simplicity, the case of a symmetric transformation ($\epsilon = \epsilon'$, $\lambda = \lambda'$) and of a period-3 attractor. Then we have

$$\Delta(\lambda, \epsilon) = \frac{\epsilon}{2} \frac{\lambda(2\lambda + 1)}{\lambda^2 - \lambda - 1}$$

and $P(n)$ takes the form

$$P(n) = \frac{\epsilon}{2\Delta(\lambda, \epsilon)} \int_C e^{sn} \frac{\lambda e^s(2\lambda e^s + 1)}{\lambda^2 e^{2s} - \lambda e^s - 1} \, ds$$

The poles of the integrand are given by

$$\lambda e^s = \begin{cases} \lambda_0 \\ -1/\lambda_0 \end{cases}$$

where $\lambda_0 = (1 + \sqrt{5}/2)$. The integration over s is trivial, since it only contains the contribution of these poles. One obtains

$$P(n) = \frac{\lambda^2 - \lambda - 1}{\lambda(2\lambda + 1)(\lambda_0 + 2)} \left[(3\lambda_0 + 2)\left(\frac{\lambda_0}{\lambda}\right)^n + (2 - \lambda_0)\left(-\frac{1}{\lambda_0 \lambda}\right)^n \right]$$

the first moment $\langle n \rangle$ being given by

$$\langle n \rangle = \sum_n P_n = \frac{(\lambda + 1)(3\lambda + 1)}{(2\lambda + 1)(\lambda^2 - \lambda - 1)} \xrightarrow[\lambda \to \lambda^0]{} \frac{\lambda_0}{\lambda - \lambda_0}$$

We notice that $P(n)$, and therefore $\langle n \rangle$, is independent of ϵ, or, equivalently, of parameter b. Varying λ in the above formulas means that we change the geometry of the trapezium. The arbitrariness in the choice of the b value is, however, limited by the fact that we demand the existence of a 3-cycle.

Now, since $\lambda > \lambda_0 > 1$, we see that $P(n) \to_{n \to \infty} (\lambda_0/\lambda)^n$, with $\lambda_0 = e^{hT}$. This result could be expected in the case of symmetric T_z, since $A = \tilde{A}$. The poles of $G(z - s)$ are the roots of $\det|\mathfrak{I} - \lambda e^s \tilde{A}| = 0$, and the largest one is given by λe^s = largest eigenvalue of $\tilde{A}$. This argument shows that the large-n behavior of $P(n)$ is general, not depending on the order k of the periodic attractor.

In the limit $\epsilon \to 0$, $\lambda \to \lambda_0$, which is the slope of the associated T_t transformation. Then $\lambda - \lambda_0 \sim \epsilon \sim \delta$ and we have

$$P(n) \underset{n \to \infty}{\sim} (1 + \alpha\delta)^{-n} \qquad (\alpha \text{ finite numerical factor}) \tag{5}$$

$$\langle n \rangle \underset{n \to \infty}{\sim} 1/\delta \tag{6}$$

It is easily seen that formulas (5) and (6) also hold for nonsymmetric T_z (in the case of a period-3 attractor one finds

$$P(n) \sim \left[\frac{1 + (1 + 4\mu)^{1/2}}{2\lambda} \right]^n$$

which takes the form (5) in the limit $\epsilon + \epsilon' \to 0$).

Expressions (5) and (6) show that, for small δ, $P(n)$ is not peaked around $\langle n \rangle$, but widely spread out. Such a type of law would be obtained in the case of a random walk in [Δ] with independent finite-amplitude jumps, and in the presence of an absorbing interval [δ]. This already suggests a stochastic behavior of the orbits in the preasymptotic region. However, it is not sufficient to characterize the dynamics in the chaotic region, where $h_T > 0$. A more specific feature of the stochastic behavior will be shown by considering a partition of [Δ] which refines the basic partition determined by the points of the periodic attractor, i.e., each element a_i of which is contained in one element of the previous partition. We ask for the probability law $P_{a_i}(n)$ associated with element a_i, that is, the probability of reaching the trapping region after n iterations, knowing that the initial point was chosen in a_i with uniform probability. An interesting partition is generated by the inverse of the fixed point p lying in [Δ]. As an example, a partition containing the five first iterates of p is shown on Fig. 5 in the case of a period-3 attractor and symmetric T_z. The matrix A corresponding to this partition is

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

It is clear that the generating function of $P_{a_i}(n)$ will be the measure $\Delta_{a_i}(\lambda, \epsilon)$ of element a_i, namely

$$\Delta_{a_i} = [\Im_{a_i}, (\Im - A/\lambda)^{-1} X_0]$$

where $\Im_{a_i} = (0, 0, \ldots, 1, 0, \ldots)$, the nonzero component being the ith.

Again, the large-n behavior of $P_{a_i}(n)$ will be associated with the largest eigenvalue of A. But the new partition is, as the previous one, a generator of topological entropy (because the inverses of any a_k all belong to an element of the partition). Therefore, the largest eigenvalue of A is still e^{h_T}, and the large-n behavior of $P_{a_i}(n)$ is the same as that of $P(n)$. This is true whatever the order N of the partition (i.e., the number N of p's inverses we use). When $N \to \infty$ the lengths of most of the a_i go to zero (not uniformly over the $\{a_i\}$ set), while the points limiting these elements get closer and closer to the unstable fixed points of the various iterates of T_z. In other words, the set of a_i boundaries converge toward the set of nonwandering points of T_z. The similarity of asymptotic laws $P_{a_i}(n)$ at large N clearly manifests the Cantor structure of this last set.

3.2. The Case of a C^2 Transformation

Let us first point out a pathological feature of the trapezoidal transformation: δ is always larger than the length $\epsilon + \epsilon'$ of the trapezium's plateau. Considering for simplicity the symmetric transformation $T_z(\lambda, \epsilon)$, and letting λ

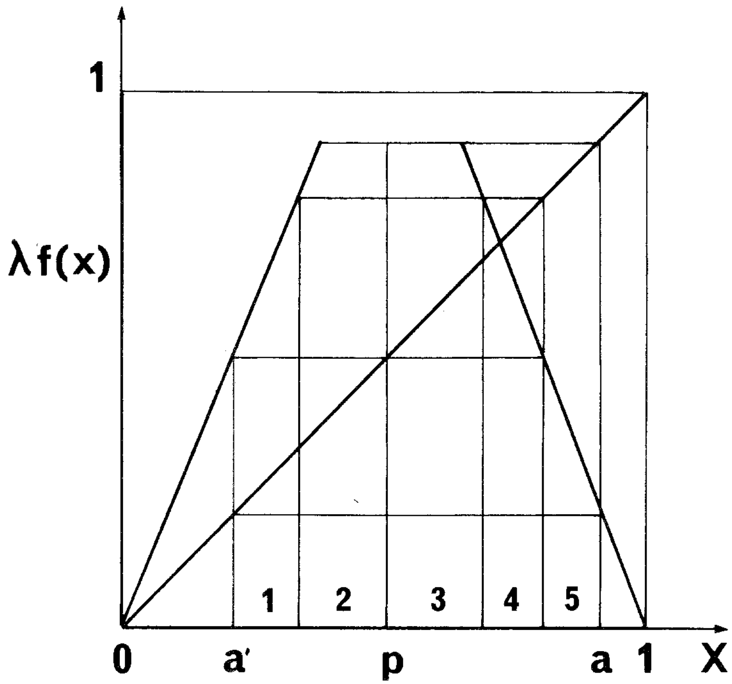

Fig. 5. Partition of the invariant interval generated by the inverses of periodic point p (case of period-3 cycle).

vary in $[\lambda_0, \lambda_{max}]$ while ϵ is fixed, we shall observe successively the various stable periodic orbits of the MSS sequence, each with a trapping domain larger than ϵ. If ϵ has not been chosen too small, all these cycles will be "conspicuous" (according May's terminology) and the average lifetime of "transitory dynamics" will be finite, whatever the period of the cycle. This situation is exceptional and does not occur if the transformation is continuously differentiable in [0, 1], provided the maximum of $T(x, \lambda)$ is not too flat. Indeed, for a cycle with high period k, the transformation is dilating in the neighborhood of most of the points of the periodic orbit. Then, the chain rule for calculating the derivatives of $T^{(k)}(x, \lambda)$ shows that the curvature of $T^{(k)}(x, \lambda)$ is in general very large in the neighborhood of $x = p$. Therefore, the width of the trapping zone is usually vanishingly small for large k, and the only observable periodic attractors will be the first ones [by "observable" we mean that they can actually be seen, taking account of the unavoidable fluctuating perturbations superimposed on the deterministic transformation either in a computer experiment or in a physical system whose evolution is modeled by $T(x, \lambda)$].

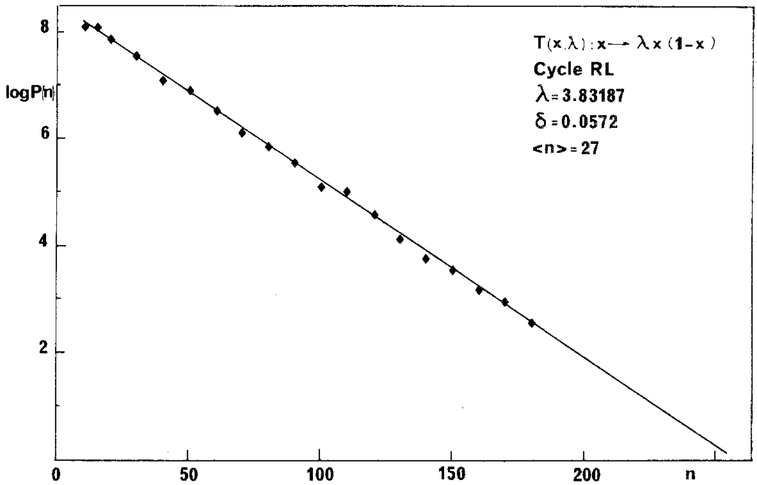

Fig. 6

In the case of differentiable transformations, we can no longer evaluate analytically the $P_{a_i}(n)$. However, it is natural to conjecture that their large-n behavior is of the same type as for the T_z transformation [properties (5), (6)]. We have verified this conjecture numerically with a good approximation on $T(x, \lambda) = \lambda x(1 - x)$ and for $\lambda = 3.6275$ (period-3 cycle) and $\lambda = 3.4057$ (period-5 cycle: RL^2R, according to the MSS notation). $P(n)$ has been determined by counting the number of iterations leading to the orbit of a point of $[\Delta]$ in the trapping interval, and by averaging over about 10^6 points in $[\Delta]$. We have verified that $\langle n \rangle$ is nearly proportional to δ^{-1} and that $\log P(n)$ varies linearly with a slope of the order of δ (see Figs. 6 and 7).

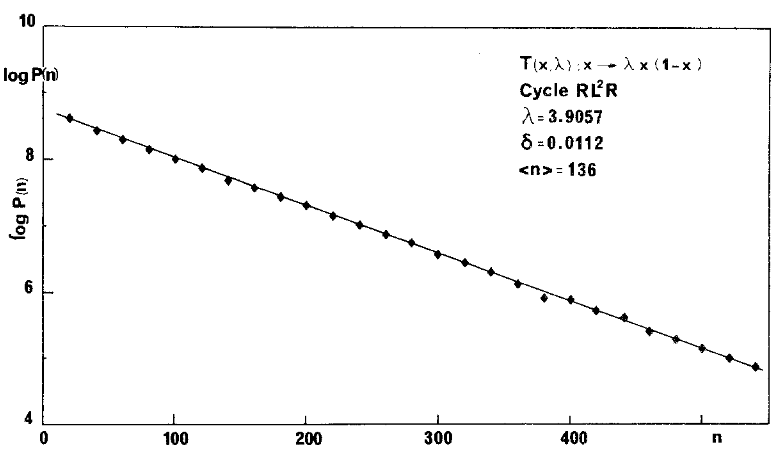

Fig. 7

APPENDIX

Theorem. Let $T(x, \tilde{\lambda})$ be a transformation of the unit interval into itself and such that (i) there exists a unique periodic attractor with a trapping interval (i.e., x_m belongs to one of the periodic point's invariant neighborhood); (ii) the following is true:

$$\left|\frac{\partial}{\partial x} f(x, \tilde{\lambda})\right| > 2 \quad \text{in} \quad \{x | x \in [0, 1]/[\delta]\}$$

Then the measure of the attractor's basin is unity.

Proof. Let us consider an interval $[h]$ in $[0, 1]$ with finite length and such that $[h] \cap [\delta] = \{0\}$. The successive images $T^{(n)}([h])$ are such that all points of $[h]$ have distinct images as long as $T^{(n)}([h])$ does not contain x_m. Since $x_m \in [\delta]$, and since the transformation is dilating for $x \in [0, 1]/[\delta]$, the lengths of successive images of $[h]$ increase with n as long as $T^{(n)}[h] \in [0, 1]/[\delta]$. Therefore, $\exists n_1$ such that $T^{(n_1)}([h] \cap [\delta] = [\xi_{n_1}] \neq \{0\}$ (eventually $[\xi_{n_1}] = [\delta]$).

Let $\{h_{n_1}\} = T^{(n_1)}([h]) - [\xi_{n_1}]$. There exists a unique interval $[a_1]$ such that $[a_1] = T^{-n_1}([\delta] \cap [h])$, and we may write $\{h_{n_1}\} = T^{(n_1)}([h] - [a_1])$. The $[a_1]$ is the set of points of $[h]$ whose images by $T^{(n_1)}$ fall in $[\delta]$.

$\{h_{n_1}\}$ is a set of intervals belonging to $[0, 1]/[\delta]$; therefore $\exists n_2$ such that

$$T^{(n_2 - n_1)}(\{h_{n_1}\}) \cap [\delta] = [\xi_{n_2}] \neq \{0\}$$

We put

$$\{h_{n_2}\} = T^{(n_2 - n_1)}(\{h_{n_1}\}) - [\xi_{n_2}] = T^{(n_2)}([h] - [a_1]) - [\xi_{n_2}]$$

There exists $[a_2]$ such that

$$\{h_{n_2}\} = T^{(n_2)}([h] - [a_1] - [a_2])$$

and so on. We obtain in this way the sets $\{n_1, n_2, \ldots, n_k\}$, $[a_1], [a_2], \ldots, [a_k]$; $\{h_{n_1}\}, \ldots, \{h_{n_k}\}$, with

$$\{h_{n_k}\} = T^{(n_k)}([h] - [a_1] - \cdots - [a_k])$$

$\{\eta_k\} = [h] - [a_1] - \cdots - [a_k]$ is the subset of $[h]$ whose orbit points do not reach $[\delta]$ after n_k iterations.

The successive images of intervals contained in $\{\eta_k\}$ may overlap. Let us consider two disjoint intervals $[q_i]$ and $[q_j]$ belonging to $T^p(\{\eta_k\})$. We may write

$$[q_i] = [a_i] + [b_i], \qquad [q_j] = [a_j] + [b_i]$$

where

$$T([a_i]) = T([a_j]) = T([q_i]) \cap T([q_j]) = [q_{ij}]$$

But

$$\text{Meas } T([q_i] + [q_j]) > q_{ij} + \alpha(b_i + b_j)$$

where

$$\alpha = \inf\{|\partial T/\partial x|: \quad x \notin [\delta]\}$$

Now if $a_i = \sup(a_i, a_j)$, we have $a_{ij} > \alpha a_i > \frac{1}{2}\alpha(a_i + a_j)$. Therefore

$$\text{Meas } T([q_i] + [q_j]) > \tfrac{1}{2}\alpha(q_i + q_j)$$

and $T^{(p+1)}(\{\eta_k\}) > \frac{1}{2}\alpha T^{(p)}(\{\eta_k\})$, from which $h_{n_k} > (\alpha/2)^{n_k}\eta_k$ or $\eta_k < (\alpha/2)^{-n_k}$ (the last inequality follows from the evident fact that $h_{n_k} < 1$). Therefore, $\gamma < (\alpha/2)^{-n_k}$. But since $\alpha/2 > 1$ (by hypothesis), $\exists n_k$ such that $(\alpha/2)^{-n_k} < \gamma$, which is contradictory. Therefore $\gamma = 0$. QED

The extension of the demonstration to the case where $f(x)$ is maximum on an interval is trivial.

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